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Solution of Convex Feasibility Problems with Block-Iteration

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Abstract

The paper presents iterative method for solving convex feasibility problems where a convex combination of projections onto the given convex sets is discussed as the weights of the combination may vary from step to step. Here some special cases are block iterative processes in each iterative step a certain subfamily of the given family of convex sets is used. Such processes will be useful in various areas of applications including image reconstruction from projections, image restoration, and other fully discretized inversion problems.

Introduction

The convex feasibility problem is to find a point in the nonempty intersection of a finite family of closed convex sets in the Euclidean space R^n . The list of real world problems modelled into such a problem is very long. It includes discretized models of image reconstruction from projections, the fully discretzed model of inverse problem in radiation therapy treatment planning and problems of image restoration. The relevant families of closed convex sets are given by either system of linear equations or inequalities or by non-linear inequalities. When derived from such real world applications the resulting convex feasibility problem is often very large, with the number of sets and higher order of magnitude. Another common feature of these problems is high sparsity. It means that the constraint matrix in the linear case or Jacobian of the convex functions is a sparse matrix.

The study derives a block iterative scheme for the convex feasibility problem which includes and therefore generalizes the two extremes of simultaneous iterations which are well-known successive-projection method for finding the common point of convex sets. We study here the convergence of the block- iterative scheme under the assumption that the convex feasibility problem is consistent, i.e. that the intersection of the sets is non empty. Interesting results have been obtained elsewhere regarding the behaviour of algorithms when applied to an inconsistent feasibility problem. Such results usually establish cyclic convergence for row-action schemes and convergence to specified points in fully simultaneous schemes.

Block-Iterative Algorithms and applications

Let $J=\{1,2,...,m\}$ and let $\{Q_j \mid j\in J\}$ be a finite family of closed convex sets in the n-dimensional Euclidean space R^n .Here $Q=\cap\{Q_j \mid j\in J\}$ is assumed to be nonempty .A function $w:J\to R^+$ is called a weight function if $\sum_{j\in J} w(j)=1$ (Here R^+ denotes the non negative ray of real).

For every $j\in J$ the orthogonal projection onto Q_j is the mapping $P_j : R^n \to Q_j \subseteq R^n$ given by $P_j(x)=\arg\min\{\|x-y\| \, y\in Q_j\}$ .For weight function we define $P_w : R^n \to R^n$

by $P_w(x)=\sum_{j\in J} w(x)P_j(x)$ .General scheme for block iterative projections can be described as:

Algorithm (A1):

Initial: $x^0 \in R^n$ is arbitrary

Iterative step:

$$x^{k+1}=x^k+\lambda_k\left[P_{w_k}\left(x^k\right)-x^k\right],$$

where $\{w_k\}$ is a fair sequence of weight functions and $\{\lambda_k\}$ is a sequence of relaxation parameters.

Algorithm (A2):

Initial: $x^0 \in R^n$ is arbitrary

Iterative step: $x^{k+1}=x^k+\lambda_k\left[\sum_{j\in J_{t(k)}} w_k(j)c_j\left(x^k\right)a^j\right]$,

where $\{t(k)\}$ is almost cyclic control sequence on $\{1,2,...,m\}$ and $c_j\left(x^k\right)$ is defined by

$$c_j\left(x^k\right)=\min\left(o,\frac{b_j-\langle a^j,x^k\rangle}{\| a^j \|^2}\right)$$

Convergence of the Block-Iterative Algorithm

The following notation will be used. For any $I\subseteq J$ and any weight function w, define $w(I)=\sum_{i\in I} w(i)$ .For any $B\subseteq R^n$ denote $I(B)=\{j\in J \mid B\cap Q_j=\phi\}$.

$B(x,r)=\{y\in R^n \mid \|x-y\|\le r\}$ is the ball with radius r centered at $x\in R^n$ .Finally, define $P_{j,\lambda}(x)=x+\lambda\left[P_j(x)-x\right]$ and $P_{w,\lambda}(x)=x+\lambda\left[P_w(x)-x\right]$,where $\lambda\in R$ and w is a weight function.

Proposition (P1): If $x\in R^n$, then for every $y\in Q_j$ and every $\lambda\in[\theta_1,2-\theta_2]$ with $\theta_1,\theta_2\succ 0$ fixed,

$$\| P_{j,\lambda}(x)-y \|\le\| x-y \|. \tag{1.1}$$

Proposition (P2): Let $q\in Q=\bigcap\{Q_j \mid j\in J\}$, $\lambda\in[\theta_1,2-\theta_2]$ with $\theta_1,\theta_2\succ 0$ fixed, and let w be a weight function. Then for every $x\in R^n$

$$\| P_{w,\lambda}(x)-q \|\le\| x-q \|. \tag{1.2}$$

Proposition (P3): Let $u\in R^n, J=I(u)$ $\lambda\in[\theta_1,2-\theta_2]$ with $\theta_1,\theta_2\succ 0$ fixed, and let w be a weight function. Then for every $\rho\succ 0$ there exists a real nonnegative γ such that if $\| x \|\le\rho$ then

$$\| P_{w,\lambda}(x)-u \|\le\| x-u \|+\gamma w(I). \tag{1.3}$$

Proposition (P4): Let $q\in Q$, $B\subseteq R^n$ a compact set, $I=I(B)$; let $\lambda\in[\theta_1,2-\theta_2]$ with $\theta_1,\theta_2\succ 0$ fixed; and let w be any weight function. Then there exists an $\alpha\succ 0$ such that for every $x\in B$

$$\| P_{w,\lambda}(x)-q \|\le\| x-q \|-\alpha w(I) . \tag{1.4}$$

Result of Theorem: If $Q \neq \phi$, if $\{w_k\}$ is any fair sequence of weight functions, and if $\{\lambda_k\}$ is any sequence of relaxation parameters for which $\lambda_k \in [\theta_1, 2-\theta_2]$ for all k=0,1,2..., where $\theta_1, \theta_2 \succ 0$, then any sequence $\{x^k\}$ generated by A1 converges to a point $x^* \in \hat{Q}$.

Proof:

Any sequence $\{x^k\}$ generated by A1 is monotone with respect to Q, i.e. for every $k = 0,1,2,...,$ $\|x^{k+1} - q\| \le \|x^k - q\|$ for any $q \in Q$. This implies that $\{x^k\}$ is bounded. Next we have to show that $\{x^k\}$ is convergent. For this let $\{x^k\}$ has two or more limit points. Let u be one of the limit points and v be the other such that $\|u - v\| = \rho$. We first show that $u \in Q$. The sequence $\{\|x^k - q\|\}$ is monotonically decreasing and bounded below. Since u is a limit point of $\{x^k\}$, it follows that $\|x^k - q\| \to \|u - q\| ask \to \infty$ $\forall k = 0,1,2...,$

$$\|x^k - q\| \ge \|u - q\| \tag{1.5}$$

Suppose that $u \notin Q$ .Choose $\tau \succ 0$ such that $\tau \prec \rho / 2$ and $B = B(u, \tau)$ satisfies $B \bigcap Q_i = \phi$ for every $i \in I(u)$. Let $I = I(B)$ and let $\gamma and \alpha$ be as in P3 andP4, respectively. Define

$$\varepsilon = \tau \frac{\alpha}{\gamma + \alpha}$$

And choose k such that $\|x^k - u\| \prec \varepsilon$ .Since v is also a limit point and $\tau \prec \rho / 2$, there exist an m$\succ$ k such that $x^m \notin B$ .Then by P4

$$\|x^m - q\| \le \|x^k - q\| - \alpha \sum_{t=k}^{m-1} w_t(I)$$
$$\prec \|u - q\| + \varepsilon - \alpha \sum_{t=k}^{m-1} w_t(I) \tag{1.6}$$

From (1.5) and (1.6), we have

$$\sum_{t=k}^{m-1} w_t(I) \prec \frac{\varepsilon}{\alpha} \tag{1.7}$$

And from P3

$$\|x^m - u\| \le \|x^k - u\| + \gamma \sum_{t=k}^{m-1} w_t(I) \tag{1.8}$$

Now from (1.7) and (1.8), We get

$$\|x^m - u\| \prec \varepsilon + \frac{\gamma\varepsilon}{\alpha} = \tau$$

contradicting , $x^m \notin B$, this gives $u \in Q$. Now by Fejer monotonocity it concludes that $\{x^k\}$ converges to u. And this done by contradiction (assuming several limit points).Hence with the help of A1 and P4 we get

$$\|x^m - q\| \le \|x^k - q\| - \alpha \sum_{t=k}^{m-1} w_t(I(B_1)) \tag{1.9}$$

For every $m \succ k$.

Hence the resultant shows convergence with the help of block iterations. The conclusion shows that such processes will be useful in various areas of applications including image reconstruction from projections, image restoration, and other fully discretized inversion problems.

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